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1. Overview

We describe here the radiation code used in our general circulation model (GCM). This code is a stand-alone version and it calculates instantaneous profiles of fluxes and heating rates given a set of basic input parameters (see below). With some modification (not included here) it could also be adopted for time marching calculations such as marching through a diurnal cycle.

The code is based on the two-stream solution to the radiative transfer equation for plane parallel atmospheres. Since computational speed is important for GCMs, we followed the approach of Toon et al. (1989) and develop a tridiagonal matrix solution for multiple layers in a vertically inhomogeneous plane parallel atmosphere that is valid for all types of two-stream equations. However, in the infrared Toon et al. show that only the hemispheric mean approximation gives physically realistic results (e.g., emissivities < 1) so we use this exclusively in this wavelength regime. In the visible, Eddington or quadrature assumptions are provided and are user selectable.

The model accounts for gaseous absorption by CO2 and water vapor, molecular Rayleigh scattering, and scattering and absorption due to water ice and dust particles. CO2 and water vapor opacities are derived from correlated-k distributions calculated off-line using a line-by-line code.

2. Two Stream Equations

In this section, we provide the user a thorough derivation of the 2-stream solutions we use in our code.

The general equation of radiative transfer in a plane parallel atmosphere is

$$\mu\frac{dI_\nu(\tau,\mu,\phi)}{d\tau} = I_\nu(\tau,\mu,\phi) - \frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} P_\nu(\mu,\mu';\phi,\phi')I_\nu(\tau,\mu,\phi)d\mu' d\phi' - S_\nu(\tau,\mu,\phi) \quad (1)$$

where μ is the cosine of the zenith angle (measured from the local vertical), $I_\nu(\tau,\mu,\phi)$ is the radiation intensity at frequency ν, τ is the optical depth, φ is the azimuth angle, ω_o is the single scattering albedo, P_ν is the scattering phase function, and S_ν is a source term.

Note that the optical depth is defined with respect to the local vertical and increases downward from the top of the atmosphere, i.e., for an absorbing atmosphere

$$\tau = -\int_z^{\infty} \rho_a k_a dz$$

(2)

where ρ_a is the absorber density and k_a is a mass absorption coefficient.

The first term on the right hand side of equation (1) represents absorption along the path. The second term represents scattering of radiation from angle μ',ϕ' into angle μ,ϕ. The phase function determines the angular distribution of scattered radiation. The integrals account for all possible scattering events within a 4π solid angle. The last term depends on which part of the spectrum solutions are sought. For visible radiation,

$$S_\nu = \frac{\omega_o}{4} F_s(\nu) P(\mu,-\mu_o,\phi,\phi_o) \exp(-\frac{\tau}{\mu_o}) \tag{3}$$

where μ_o is the cosine of the solar zenith angle. In the infrared,

$$S_\nu = (1-\omega_o) B_\nu(T) \tag{4}$$

where $B_\nu(T)$ is the Planck function at temperature T. Equation (3) represents that part of the extinguished direct solar beam coming from angle $-\mu_o$ that is singly scattered into the direction μ,ϕ. Equation (4) represents the contribution due to thermal emission.

Our main interest is obtaining reasonably accurate heating rates. We do not need to know the angular dependence of the radiation field. Hence, we seek methods that will give us the upward and downward fluxes (from which heating rates can be derived). Two-stream solutions to equation (1) give us such fluxes. They reduce the angular dependence of the intensity field to two streams: up and down. Generalized radiative transfer equations for these up and down streams can be obtained by integrating equation (1) separately over the upper (μ=0 to 1) and lower hemispheres (μ=0 to -1), and for all azimuth angles (ϕ=0 to 2π). Noting that the up and down fluxes are defined by

$$F^+ = \int_0^{2\pi}\int_0^1 I_\nu \mu d\mu d\phi$$

(5)

and

$$F^- = \int_0^{2\pi}\int_0^{-1} I_\nu \mu d\mu d\phi$$

(6)

respectively, equation (1) becomes

$$\frac{dF^+}{d\tau} = \int_0^{2\pi}\int_0^1 I_\nu d\mu d\phi - \int_0^{2\pi}\int_0^1 [\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^1 P_\nu(\mu,\mu';\phi,\phi') I_\nu d\mu' d\phi'] d\mu d\phi - \int_0^{2\pi}\int_0^1 S_\nu d\mu d\phi \tag{7}$$

for the upward flux F^+ and

$$\frac{dF^-}{d\tau} = +\int_0^{2\pi}\int_0^{-1} I_\nu d\mu d\phi - \int_0^{2\pi}\int_0^{-1}[\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P_\nu(\mu,\mu^{'};\phi,\phi^{'})I_\nu d\mu^{'} d\phi^{'}]d\mu d\phi - \int_0^{2\pi}\int_0^{-1} S_\nu d\mu d\phi \tag{8}$$

for the downward flux F^-.

Solutions to equation (7) and (8) depend on the assumptions made about the angular dependence of the phase function and intensity field. Here, we give sample solutions for the two types of approximations we commonly use: the Eddington approximation for solar radiation, and the hemispheric mean approximation for infrared radiation. With these approximations, equation (7) and (8) will be shown to reduce to two first-order coupled linear differential equations that can be solved using standard methods. Other approximations lead to a similar set of equations but with different coefficients.

The Eddington Approximation

In the Eddington approximation, the intensity is expressed as

$$I = I_o + \mu I_1 \tag{9}$$

where I_0 and I_1 are constants (we now drop the frequency notation). The phase function is approximated by a second order Legendre polynomial and takes the form

$$P(\Theta) = 1 + \omega_1 \cos(\Theta) \tag{10}$$

where Θ is the scattering angle (i.e., the angle between μ,ϕ and $\mu^{'},\phi^{'}$). In our μ,ϕ notation this becomes

$$P(\mu,\mu^{'};\phi,\phi^{'}) = 1 + \omega_1[\mu\mu^{'} + (1-\mu^2)(1-\mu^{'2})\cos(\phi-\phi^{'})] \tag{11}$$

We now systematically evaluate the three terms on the right hand side of equation (7) on the basis of these approximations. The first term is straightforward.

$$\int_0^{2\pi}\int_0^{1} I d\mu d\phi = 2\pi(I_o + \tfrac{1}{2}I_1) \tag{12}$$

The second term takes a bit more work. First evaluate the inside double integral.

$$\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\mu';\phi,\phi') I d\mu' d\phi' = \frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\{1+\omega_1[\mu\mu' + (1-\mu^2)(1-\mu'^2)\cos(\phi-\phi')]\}(I_o+\mu' I_1)d\mu' d\phi'$$

$$= \frac{\omega_o}{2}\int_{-1}^{1}(1+\omega_1\mu\mu')I_o d\mu' + \frac{\omega_o}{2}\int_{-1}^{1}(1+\omega_1\mu\mu')\mu' I_1 d\mu' + \ldots\cos(\phi-\phi')terms$$

$$= \frac{\omega_o}{2}(2I_o+0) + \frac{\omega_o}{2}(0+\tfrac{2}{3}\omega_1\mu) \qquad (13)$$

$$= \omega_o I_o + \tfrac{1}{3}\omega_o\omega_1\mu I_1$$

$$= \omega_o(I_o + g\mu I_1)$$

Note here that μ’ is used in the Eddington approximation (i.e., $I=I_o+\mu' I_1$), that all terms involving the $\cos(\phi-\phi_o)$ integrate out to zero so they can be dropped immediately, and g is defined as $g=\omega_1/3$.

Now, taking the outside double integrals we have

$$\int_0^{2\pi}\int_0^{1}\left[\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\mu';\phi,\phi')I_v d\mu' d\phi'\right]d\mu d\phi = \int_0^{2\pi}\int_0^{1}\omega_o(I_o+g\mu I_1)d\mu d\phi$$

$$= 2\pi(\omega_o I_o + \tfrac{1}{2}\omega_o g I_1)$$

(14)

For the third term, the integration of the source function (using equation (3)) gives

$$\int_0^{2\pi}\int_0^{1} S d\mu d\phi = \int_0^{2\pi}\int_0^{1}\frac{\omega_o}{4}F_s P(\mu,-\mu_o,\phi,\phi_o)\exp(-\frac{\tau}{\mu_o})d\mu d\phi$$

$$= \frac{\omega_o}{4}F_s\exp(-\frac{\tau}{\mu_o})\int_0^{2\pi}\int_0^{1} 1+\omega_1[-\mu\mu_o + (1-\mu^2)(1-\mu_o^2)\cos(\phi-\phi_o)]d\mu d\phi$$

$$= \frac{\omega_o}{4}F_s\exp(-\frac{\tau}{\mu_o})[2\pi - \int_0^{2\pi}\int_0^{1}\omega_1\mu\mu_o d\mu d\phi + \ldots.\cos(\phi-\phi_o)terms] \qquad (15)$$

$$= \frac{\omega_o}{4}F_s\exp(-\frac{\tau}{\mu_o})(2\pi - 2\pi\frac{\omega_1\mu_o}{2})$$

$$= \frac{\omega_o}{2}\pi F_s\exp(-\frac{\tau}{\mu_o})(1-\frac{\omega_1\mu_o}{2})$$

$$= \frac{(2-3g\mu_o)}{4}\pi F_s\omega_o\exp(-\frac{\tau}{\mu_o})$$

Note here that we have replaced μ’ with μ_0 when using equation (11).

Equation (7) now becomes

$$\frac{dF^+}{d\tau} = 2\pi(I_o + \mu I_1) + \omega_o(I_o + g\mu I_1) + \frac{(2 - 3g\mu_o)}{4}\pi F_s(\nu)\omega_o \exp(-\frac{\tau}{\mu_o})$$
(16)

The final step is to write this equation in terms of fluxes. In the Eddington approximation the up and down fluxes are

$$F^+ = 2\pi\int_0^1 I\mu d\mu = 2\pi\int_0^1 (I_o + \mu I_1)\mu d\mu = \pi(I_0 + \tfrac{2}{3} I_1) \quad (17)$$

and

$$F^- = 2\pi\int_0^{-1} I\mu d\mu = 2\pi\int_0^{-1} (I_o + \mu I_1)\mu d\mu = \pi(I_0 - \tfrac{2}{3} I_1) \quad (18)$$

Using these definitions and noting the following additive properties

$$F^+ + F^- = 2\pi I_o \quad and \quad F^+ - F^- = \tfrac{4}{3}\pi I_1$$
(19)

equation (16) can be re-written as

$$\frac{dF^+}{d\tau} = [\frac{7 - \omega_o(4 + 3g)}{4}]F^+ + [\frac{1 + \omega_o(4 - 3g)}{4}]F^- + \frac{(2 - 3g\mu_o)}{4}\pi F_s(\nu)\omega_o \exp(-\frac{\tau}{\mu_o}) \quad (20)$$

Going through a similar procedure for the downward flux, equation (8) can be shown to reduce to

$$\frac{dF^-}{d\tau} = -[\frac{1 - \omega_o(4 - 3g)}{4}]F^+ - [\frac{7 - \omega_o(4 + 3g)}{4}]F^- + \frac{(2 + 3g\mu_o)}{4}\pi F_s(\nu)\omega_o \exp(-\frac{\tau}{\mu_o}) \quad (21)$$

The Hemispheric Mean Approximation

In the Hemispheric Mean approximation, the intensity is constant in each hemisphere such that

$$I = I^+ \quad , \quad \mu > 0$$
(22)

and

$$I = I^- \quad , \quad \mu < 0 \quad (23)$$

The phase function is symmetric and is approximated as follows:

$$P(\mu,\mu^{'}) = (1+g), \quad \mu\mu^{'} > 0 \quad (i.e., \; the \; forward \; hemisphere) \tag{24}$$

and

$$P(\mu,\mu^{'}) = (1-g), \quad \mu\mu^{'} < 0 \quad (i.e., \; the \; backward \; hemisphere) \tag{25}$$

where g is the asymmetry factor. For g = 1 all the radiation is scattered into the forward hemisphere; for g=0 the radiation is scattered equally into each hemisphere (isotropic); and for g=-1, all the radiation is scattered into the backward hemisphere.

As before, we evaluate each term on the right hand side of equation (7). Again, the first term is straightforward.

$$\int_0^{2\pi}\int_0^1 I d\mu d\phi = 2\pi I^{+} \tag{26}$$

The inside double integral of the second term is

$$\begin{aligned}\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\mu^{'};\phi,\phi^{'}) I d\mu^{'} d\phi^{'} &= \frac{\omega_o}{4\pi}\{2\pi[\int_{-1}^{0}(1-g)I^{-}d\mu^{'} + \int_0^1 (1+g)I^{+}d\mu^{'}] \\ &= \frac{\omega_o}{2}[(1-g)I^{-} + (1+g)I^{+}]\end{aligned} \tag{27}$$

So that the outside double integrals become

$$\begin{aligned}\int_0^{2\pi}\int_0^1 [\frac{\omega_o}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\mu^{'};\phi,\phi^{'}) I_\nu d\mu^{'} d\phi^{'}] d\mu d\phi &= \int_0^{2\pi}\int_0^1 \frac{\omega_o}{2}[(1-g)I^{-} + (1+g)I^{+}] d\mu d\phi \\ &= \pi\omega_o[(1-g)I^{-} + (1+g)I^{+}]\end{aligned} \tag{28}$$

The thermal infrared source function (the third term) is straightforward.

$$\begin{aligned}\int_0^{2\pi}\int_0^1 S d\mu d\phi &= \int_0^{2\pi}\int_0^1 (1-\omega_o) B d\mu d\phi \\ &= 2\pi(1-\omega_o)B\end{aligned}$$

(29)

Equation (7) for the Hemispheric Mean approximation with an infrared source term now becomes

$$\frac{dF^{+}}{d\tau} = 2\pi I^{+} - \pi\omega_o[(1-g)I^{-} + (1+g)I^{+}] - 2\pi(1-\omega_o)B \tag{30}$$

which can be written in terms of fluxes ($F^+ = 2\pi I^+$) as

$$\frac{dF^+}{d\tau} = F^+ - \frac{\omega_o}{2}[(1-g)F^- + (1+g)F^+] - 2\pi(1-\omega_o)B \tag{31}$$

For the downward flux, equation (8) for the hemispheric mean approximation reduces to

$$\frac{dF^-}{d\tau} = F^- + \frac{\omega_o}{2}[(1-g)F^- + (1+g)F^+] + 2\pi(1-\omega_o)B \tag{32}$$

3. <u>Generalized Solution for a Homogeneous Atmosphere</u>

Notice that equations (20)-(21) and (31)-(32) have a common form. Meador and Weaver (1980) showed that all two-stream equations can be written as

$$\frac{dF^\uparrow}{d\tau} = \gamma_1 F^\uparrow - \gamma_2 F^\downarrow - S^\uparrow \tag{33}$$

$$\frac{dF^\downarrow}{d\tau} = \gamma_2 F^\uparrow - \gamma_1 F^\downarrow + S^\downarrow \tag{34}$$

where the gamma coefficients depend on the type of two-stream approximation employed, and the source terms are

$$S^\uparrow = \gamma_3 \pi F_s \omega_o \exp(-\frac{\tau}{\mu_o}),\ and\ S^\downarrow = \gamma_4 \pi F_s \omega_o \exp(-\frac{\tau}{\mu_o}) \tag{35}$$

for solar radiation, and

$$S^\uparrow = S^\downarrow = 2\pi(1-\omega_o)B(\tau) \tag{36}$$

for infrared radiation.

Equations (22) and (23) are a set of coupled linear differential equations that can be solved by differentiating each with respect to τ and reordering the terms to arrive at the following two equations:

$$\frac{\partial^2 F^\uparrow}{\partial\tau^2} - \lambda^2 F^\uparrow = -(\gamma_1 S^\uparrow + \gamma_2 S^\downarrow + \frac{\partial S^\uparrow}{\partial\tau})$$

(37) (26)

$$\frac{\partial^2 F^\downarrow}{\partial\tau^2} - \lambda^2 F^\downarrow = -(\gamma_2 S^\uparrow + \gamma_1 S^\downarrow - \frac{\partial S^\downarrow}{\partial\tau}) \tag{38}$$

where

$$\lambda^2 \equiv \gamma_1^2 - \gamma_2^2 \tag{39}$$

The solution to these equations is *(Chris: I can't quite get this solution. Not sure where Γ comes from)*.

$$F^{\uparrow}(\tau) = k_1 \exp(\lambda\tau) + k_2 \Gamma \exp(-\lambda\tau) + C^{\uparrow}(\tau) \tag{40}$$

$$F^{\downarrow}(\tau) = k_1 \Gamma \exp(\lambda\tau) + k_2 \exp(-\lambda\tau) + C^{\downarrow}(\tau) \tag{41}$$

where k_1 and k_2 are constants determined by the boundary conditions, and

$$\Gamma \equiv \frac{\gamma_2}{\gamma_1 + \lambda} = \frac{\gamma_1 - \lambda}{\gamma_2}$$
(42)

and so it depends on the nature of the two stream approximation.

The first two terms on the right hand side of equations (29) and (30) represent the homogenous solution, and the C terms represent the particular solution, which depends on the source function. For solar radiation the source terms are given by equation (24) and the C terms can be shown to be

$$C^{\uparrow}(\tau) = \pi F_s \omega_o \exp(-\frac{\tau}{\mu_o}) \frac{[\gamma_3(\gamma_1 - \frac{1}{\mu_o}) + \gamma_4\gamma_2]}{(\lambda^2 - \frac{1}{\mu_o^2})} \tag{43}$$

$$C^{\downarrow}(\tau) = \pi F_s \omega_o \exp(-\frac{\tau}{\mu_o}) \frac{[\gamma_4(\gamma_1 + \frac{1}{\mu_o}) + \gamma_3\gamma_2]}{(\lambda^2 - \frac{1}{\mu_o^2})} \tag{44}$$

For thermal radiation, the source terms are given by equation (25). However, before the C terms can be evaluated the dependence of the Planck function on optical depth needs to be specified. We assume that the Planck function is linear in τ, or equivalently is approximated by the first two terms in a Taylor expansion. Thus,

$$B(\tau) = B_o + B_1\tau \tag{45}$$

where B_o is the Planck function evaluated at the top of the layer (τ=0), and B_1 is approximated as

$$B_1 = \frac{\partial B}{\partial \tau} \cong \frac{B(T_{bot}) - B_0}{\tau_*}$$

where T_{bot} is the temperature at the bottom of the layer,nd τ_* is the optical depth of the layer. This assumption is valid only for small changes in the optical depth.

Substituting equation (25) into (26) and (27) and making use of (34), the C terms for a thermal source function can be shown to be

$$C^{\uparrow}(\tau) = 2\pi(\frac{1-\omega_o}{\gamma_1 - \gamma_2})[B_o + B_1(\tau + \frac{1}{\gamma_1 + \gamma_2})] \tag{46}$$

$$C^{\downarrow}(\tau) = 2\pi(\frac{1-\omega_o}{\gamma_1 - \gamma_2})[B_o + B_1(\tau - \frac{1}{\gamma_1 + \gamma_2})] \tag{47}$$

As mentioned above, there are a variety of two stream solutions to the radiative transfer equation that depend on the assumptions made about the intensity field and phase function. Table 1 gives the approximations made for the two-stream solutions available in our code, along with their corresponding coefficients.

Table 1: Two-Stream Parameters. Note that $\gamma_4 = 1 - \gamma_3$

Method	Intensity Field	Phase Function	γ_1	γ_2	γ_3
Eddington	$I = I_o + \mu I_1$	$P(\Theta) = 1 + \omega_1 \cos(\Theta)$	$\frac{7 - \omega_o(4+3g)}{4}$	$\frac{-[1 - \omega_o(4-3g)]}{4}$	$\frac{2 - 3g\mu_o}{4}$
Quadrature	$I^+, \mu > 0$ $I^-, \mu < 0$	$P(\mu) = 1 + \sqrt{3}\mu$	$\frac{\sqrt{3}}{2}[2 - \omega_o(1+g)]$	$\frac{\sqrt{3}}{2}\omega_o(1-g)$	$\frac{1 - \sqrt{3}g\mu_o}{2}$
Hem. Mean	$I^+, \mu > 0$ $I^-, \mu < 0$	$P = 1 + g, forward$ $= 1 - g, backward$	$2 - \omega_o(1+g)$	$\omega_o(1-g)$	Not needed in IR

We emphasize here that in the infrared, only the hemispheric mean approximation conserves energy (see Toon et al., 1989). Thus, we use this exclusively in this wavelength regime.

Typical boundary conditions at visible wavelengths have

$$F^{\downarrow}(0) = 0$$

$$F^{\uparrow}(\tau_*) = A[F^{\downarrow}(\tau_*) + \pi F_o \exp(-\frac{\tau_*}{\mu_o})]$$

while in the infrared they are

$$F^{\downarrow}(0) = 0$$

$$F^{\uparrow}(\tau_*) = \varepsilon\pi B_g + (1-\varepsilon)F^{\downarrow}(\tau_*)$$

where A is the surface albedo, ε is the surface emissivity, and B_g is the Planck function evaluated at the temperature of the ground.

4. Application to Multiple Layers

Equations (29) and (30) can be applied to vertically inhomogeneous atmospheres by dividing the atmosphere into N homogenous layers. Application of boundary conditions then leads to a matrix equation, which can be inverted to obtain the fluxes. Our code follows closely the work of Toon et al (1989) is setting up this matrix. Figure 2 illustrates the concept.

Fig. 2. An N layer atmosphere. Note the distinction between level and layer indicies.

Here we define the optical depth in each layer n to range from 0 at the top of the layer, to τ_n at the bottom of the layer. In general, τ_n varies from layer to layer. The total column optical depth at any level n is just the sum of the layer optical depths

$$(\tau_c)_n = \sum_{l=1}^{n} \tau_l \quad (48)$$

The two stream solutions for a given layer n can then be written as

$$F_n^{\uparrow}(\tau) = k_{1,n} \exp(\lambda_n \tau) + k_{2,n} \Gamma_n \exp(-\lambda_n \tau) + C_n^{\uparrow}(\tau) \quad (49)$$

$$F_n^{\downarrow}(\tau) = k_{1,n} \Gamma_n \exp(\lambda_n \tau) + k_{2,n} \exp(-\lambda_n \tau) + C_n^{\downarrow}(\tau) \quad (50)$$

Before proceeding further, note that when ω_o=0 (pure absorption) the Γ_n=0 and equation (49) and (50) become decoupled, i.e., they have no common variables. Physically, the upwelling and downwelling streams experience no change of direction due to scattering so the fluxes do not depend on each other. This means that any matrix solution will have numerical difficulties in this limit. To get around this problem we introduce new variables Y_1 and Y_2 such that

$$Y_{1,n} = \frac{k_{1,n} + k_{2,n}}{2}$$
(51)

$$Y_{2,n} = \frac{k_{1,n} - k_{2,n}}{2} \quad (52)$$

Hence,

$$k_{1,n} = Y_{1,n} + Y_{2,n} \quad and \quad k_{2,n} = Y_{1,n} - Y_{2,n} \quad (53)$$

and equations (49) and (50) become

$$F_n^{\uparrow}(\tau) = Y_{1,n}[\exp(\lambda_n \tau) + \Gamma_n \exp(-\lambda_n \tau)] + Y_{2,n}[\exp(\lambda_n \tau) - \Gamma_n \exp(-\lambda_n \tau)] + C_n^{\uparrow}(\tau)$$
(54)

$$F_n^{\downarrow}(\tau) = Y_{1,n}[\Gamma_n \exp(\lambda_n \tau) + \exp(-\lambda_n \tau)] + Y_{2,n}[\Gamma_n \exp(\lambda_n \tau) - \exp(-\lambda_n \tau)] + C_n^{\downarrow}(\tau) \tag{55}$$

Now, when the Γ_n=0 the equations remain coupled.

We now proceed to set up the matrix equation. This matrix comes from the boundary conditions (see Figure 2). At the top of the atmosphere we allow for a downward diffusive flux (to account for the 3 degree background radiation, say for example). At the bottom of the atmosphere the surface can emit thermal radiation and reflect the downward flux (visible or infrared). In the interior of the atmosphere we require the downward (upward) flux at the bottom of layer n be equal to the downward (upward) flux at the top of layer n+1. Mathematically these boundary conditions are

$$F_1^{\downarrow}(\tau = 0) = F_0^{\downarrow} \tag{56}$$

$$F_n^{\downarrow}(\tau = \tau_n) = F_{n+1}^{\downarrow}(\tau = 0),\ \mathit{for}\ n = 1, N-1 \tag{57}$$

$$F_n^{\uparrow}(\tau = \tau_n) = F_{n+1}^{\uparrow}(\tau = 0),\ \mathit{for}\ n = 1, N-1 \tag{58}$$

$$F_N^{\uparrow}(\tau = \tau_n) = B_*^{\uparrow} + \mathrm{Ref} * F_N^{\downarrow}(\tau = \tau_N) \tag{59}$$

Application of these leads to

$$Y_{1,1}(\Gamma_1 + 1) + Y_{2,1}(\Gamma_1 - 1) = F_0^{\downarrow} \tag{60}$$

$$\begin{aligned} &Y_{1,n}[\Gamma_n \exp(\lambda_n \tau_n) + \exp(-\lambda_n \tau_n)] + Y_{2,n}[\Gamma_n \exp(\lambda_n \tau_n) - \exp(-\lambda_n \tau_n)] + C_n^{\downarrow}(\tau_n) = \\ &Y_{1,n+1}(\Gamma_{n+1} + 1) + Y_{2,n+1}(\Gamma_{n+1} - 1) + C_{n+1}^{\downarrow}\ ,\ \mathit{for}\ n = 1, N-1 \end{aligned} \tag{61}$$

$$\begin{aligned} &Y_{1,n}[\exp(\lambda_n \tau_n) + \Gamma_n \exp(-\lambda_n \tau_n)] + Y_{2,n}[\exp(\lambda_n \tau_n) - \Gamma_n \exp(-\lambda_n \tau_n)] + C_n^{\uparrow}(\tau_n) = \\ &Y_{1,n+1}(\Gamma_{n+1} + 1) + Y_{2,n+1}(1 - \Gamma_{n+1}) + C_{n+1}^{\uparrow}\ ,\ \mathit{for}\ n = 1, N-1 \end{aligned} \tag{62}$$

$$\begin{aligned} &Y_{1,N}[\exp(\lambda_N \tau_N) + \Gamma_N \exp(-\lambda_N \tau_N) + \\ &Y_{2,N}[\exp(\lambda_N \tau_N) - \Gamma_N \exp(-\lambda_N \tau_N)] + C_N^{\uparrow}(\tau_N) = \\ &B_*^{\uparrow} + \mathrm{Ref} * \begin{Bmatrix} Y_{1,N}[\Gamma_N \exp(\lambda_N \tau_N) + \exp(-\lambda_N \tau_N)] + \\ Y_{2,N}[\Gamma_N \exp(\lambda_N \tau_N) - \exp(-\lambda_N \tau_N)] + C_N^{\downarrow}(\tau_N) \end{Bmatrix} \end{aligned} \tag{63}$$

Equations (60)-(63) consist of 2N equations in 2N unknowns (the Y's or equivalently the k's). They can be rearranged to form a penta-diagonal matrix equation, which can be solved using standard matrix inversion techniques. However, if we can tri-diagonalize the matrix we can greatly speed up the solution. Before doing this, lets simplify the notation. If we define the following variables:

$$e_{1,n}(\tau_n) = \exp(\lambda_n \tau_n) + \Gamma_n \exp(-\lambda_n \tau_n) \tag{64}$$

$$e_{2,n}(\tau_n) = \exp(\lambda_n \tau_n) - \Gamma_n \exp(-\lambda_n \tau_n) \tag{65}$$

$$e_{3,n}(\tau_n) = \Gamma_n \exp(\lambda_n \tau_n) + \exp(-\lambda_n \tau_n) \tag{66}$$

$$e_{4,n}(\tau_n) = \Gamma_n \exp(\lambda_n \tau_n) - \exp(-\lambda_n \tau_n) \tag{67}$$

Then equations (61) and (62) become

$$Y_{1,n} e_{3,n}(\tau_n) + Y_{2,n} e_{4,n}(\tau_n) + C_n^{\downarrow}(\tau_n) = Y_{1,n+1} e_{3,n+1}(0) + Y_{2,n+1} e_{4,n+1}(0) + C_{n+1}^{\downarrow}\ ,\quad for\ n = 1, N-1 \tag{68}$$

$$Y_{1,n} e_{1,n}(\tau_n) + Y_{2,n} e_{2,n}(\tau_n) + C_n^{\uparrow}(\tau_n) = Y_{1,n+1} e_{1,n+1}(0) + Y_{2,n+1} e_{2,n+1}(0) + C_{n+1}^{\uparrow}\ ,\quad for\ n = 1, N-1 \tag{69}$$

To tridiagonalize the matrix we perform the following operations:

$$e_{4,n+1}(0) * Equation(59) - e_{2,n+1}(0) * Equation(58)$$

and

$$e_{3,n+1}(\tau_n) * Equation(59) - e_{2,n+1}(\tau_n) * Equation(58)$$

After considerable manipulation, we arrive at the following equations:

$$\begin{aligned} &Y_{1,n}\ [e_{1,n}(\tau_n)\ * e_{4,n+1}(0) - e_{3,n}(\tau_n)\ * e_{2,n+1}(0)]\ + \\ &Y_{2,n}\ [e_{2,n}(\tau_n)\ * e_{4,n+1}(0) - e_{4,n}(\tau_n)\ * e_{2,n+1}(0)]\ + \\ &Y_{1,n+1}[e_{3,n+1}(0)\ * e_{2,n+1}(0) - e_{1,n+1}(0)\ * e_{4,n+1}(0)] = e_{4,n+1}(0)[C_{n+1}^{\uparrow}(0)\ -\ C_n^{\uparrow}(\tau_n)] + \\ &\qquad\qquad e_{2,n+1}(0)[C_{n+1}^{\downarrow}(\tau_n) -\ C_{n+1}^{\downarrow}(0)] \end{aligned} \tag{70}$$

and

$$
\begin{aligned}
&Y_{2,n} \quad [e_{2,n}(\tau_n) \ * e_{3,n}(\tau_n) - e_{4,n}(\tau_n) \ * e_{1,n}(\tau_n)] \ + \\
&Y_{1,n+1} \ [e_{3,n+1}(0) \ * e_{1,n}(\tau_n) - e_{1,n+1}(0) \ * e_{3,n}(\tau_n)] \ + \\
&Y_{2,n+1} \ [e_{4,n+1}(0) \ * e_{1,n}(\tau_n) - e_{2,n+1}(0) * \ e_{3,n}(\tau_n)] = e_{3,n}(\tau_n)[C^{\uparrow}_{n+1}(0) - C^{\uparrow}_{n}(\tau_n)] + \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e_{1,n}(\tau_n)[C^{\downarrow}_{n}(\tau_n) - C^{\downarrow}_{n+1}(0)]
\end{aligned}
\tag{71}
$$

If we simplify the Y notation by defining new variables such that

$$
\begin{aligned}
&Y_1 \equiv Y_{1,1} \\
&Y_2 \equiv Y_{2,1} \\
&Y_3 \equiv Y_{1,2} \\
&Y_4 \equiv Y_{2,2} \\
&Y_5 \equiv Y_{1,3} \\
&Y_6 \equiv Y_{2,3} \\
&\vdots
\end{aligned}
$$

Then in general

$$Y_l = Y_{1,1}, Y_{1,2}, Y_{1,3}, Y_{1,4}, \ldots. Y_{1,n} \quad for\, l\, odd$$

and

$$Y_l = Y_{2,1}, Y_{2,2}, Y_{2,3}, Y_{2,4}, \ldots. Y_{2,n} \quad for\, l\, even$$

Equations (70) and (71) are now are in the tridiagonal matrix form

$$AY_{l-1} + BY_l + CY_{l+1} = D \tag{72}$$

where for l =2,2N-2,2 (i.e., l even)

$$
\begin{aligned}
A_l &= [e_{1,n}(\tau_n)e_{4,n}(0) - e_{3,n}(\tau_n)e_{2,n+1}(0)] \\
&= e_{1,n}(\Gamma_{n+1} - 1) - e_{3,n}(1 - \Gamma_{n+1}) \\
&= (e_{1,n} + e_{3,n})(\Gamma_{n+1} - 1)
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
B_l &= [e_{2,n}(\tau_n)e_{4,n+1}(0) - e_{4,n}(\tau_n)e_{2,n+1}(0)] \\
&= e_{2,n}(\Gamma_{n+1} - 1) - e_{4,n}(1 - \Gamma_{n+1}) \\
&= (e_{2,n} + e_{4,n})(\Gamma_{n+1} - 1)
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
C_l &= [e_{3,n+1}(0)e_{2,n+1}(0) - e_{1,n+1}(0)e_{4,n+1}(0)] \\
&= [(\Gamma_{n+1}+1)(1-\Gamma_{n+1}) - (1+\Gamma_{n+1})(\Gamma_{n+1}-1)] \\
&= 2(1-\Gamma_{n+1}^2)
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
D_l &= e_{4,n+1}(0)[C_{n+1}^{\uparrow}(0) - C_n^{\uparrow}(\tau_n)] + e_{2,n+1}(0)[C_n^{\downarrow}(\tau_n) - C_{n+1}^{\downarrow}(0)] \\
&= (\Gamma_{n+1}-1)[C_{n+1}^{\uparrow}(0) - C_n^{\uparrow}(\tau_n)] + (1-\Gamma_{n+1})[C_n^{\downarrow}(\tau_n) - C_{n+1}^{\downarrow}(0)]
\end{aligned} \tag{76}
$$

and for l=3,2N-1,2 (i.e., l odd)

$$
\begin{aligned}
A_l &= [e_{2,n}(\tau_n)e_{3,n}(\tau_n) - e_{4,n}(\tau_n)e_{1,n}(\tau_n)] \\
&= [\exp(\lambda_n\tau_n) - \Gamma_n \exp(-\lambda_n\tau_n)][\Gamma_n \exp(\lambda_n\tau_n) + \exp(-\lambda_n\tau_n)] - \\
&\quad [\Gamma_n \exp(\lambda_n\tau_n) - \exp(-\lambda_n\tau_n)][\exp(\lambda_n\tau_n) + \Gamma_n \exp(-\lambda_n\tau_n)] \\
&= [\Gamma_n \exp(2\lambda_n\tau_n) + 1 - \Gamma_n^2 - \Gamma_n \exp(-2\lambda_n\tau_n)] - \\
&\quad [\Gamma_n \exp(2\lambda_n\tau_n) + \Gamma_n^2 - 1 - \Gamma_n \exp(-2\lambda_n\tau_n)] \\
&= 2(1-\Gamma_n^2)
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
B_l &= [e_{3,n+1}(0)e_{1,n}(\tau_n) - e_{1,n+1}(0)e_{3,n}(\tau_n)] \\
&= [(\Gamma_{n+1}+1)e_{1,n}(\tau_n) - (1+\Gamma_{n+1})e_{3,n}(\tau_n)] \\
&= (1+\Gamma_{n+1})[e_{1,n}(\tau_n) - e_{3,n}(\tau_n)]
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
C_l &= e_{4,n+1}(0)e_{1,n}(\tau_n) - e_{2,n+1}(0)e_{3,n}(\tau_n) \\
&= (\Gamma_{n+1}-1)e_{1,n}(\tau_n) - (1-\Gamma_{n+1})e_{3,n}(\tau_n) \\
&= [e_{1,n}(\tau_n) + e_{3,n}(\tau_n)](\Gamma_{n+1}-1)
\end{aligned} \tag{79}
$$

$$
D_l = e_{3,n}(\tau_n)[C_{n+1}^{\uparrow}(0) - C_n^{\uparrow}(\tau_n)] + e_{1,n}(\tau_n)[C_n^{\downarrow}(\tau_n) - C_{n+1}^{\downarrow}(0)] \tag{80}
$$

For l=1 (the top layer) comparing equation (60) with (72) we see by inspection that

$$
A_l = 0 \tag{81}
$$

$$
B_1 = \Gamma_1 + 1 \tag{82}
$$

$$
C_1 = \Gamma_1 - 1 \tag{83}
$$

$$
D_1 = F_o^{\downarrow} - C_1^{\downarrow}(0) \tag{84}
$$

For l=L (the bottom layer) comparing equation (63) with (72) we can see that

$$A_L = \exp(\lambda_n \tau_n) + \Gamma_n \exp(-\lambda_n \tau_n) - \mathrm{Ref}[\Gamma_n \exp(\lambda_n \tau_n) + \exp(-\lambda_n \tau_n)]$$
$$= (1 - \mathrm{Ref}\,\Gamma_n)\exp(\lambda_n \tau_n) + (\Gamma_n - \mathrm{Ref})\exp(-\lambda_n \tau_n) \quad (85)$$

$$B_L = \exp(\lambda_n \tau_n) - \Gamma_n \exp(-\lambda_n \tau_n) - \mathrm{Ref}[\Gamma_n \exp(\lambda_n \tau_n) - \exp(-\lambda_n \tau_n)]$$
(86)

$$C_L = 0 \quad (87)$$

$$D_L = B_*^{\uparrow} - C_N^{\uparrow}(\tau_n) + \mathrm{Ref}\, C_N^{\downarrow}(\tau_N)$$
(88)

5. <u>Additional Sophistication: The Delta Approach</u>

Aerosol phase functions generally have a strong forward peak, which is not well captured by equation (10). Replacing this phase function with the sum of a delta function in the forward direction and a less sharply peaked remainder in the backward direction leads to higher accuracy in the 2-stream approximation. Thus,

$$P_\delta(\cos\Theta) \cong 2f\delta(1 - \cos\Theta) + (1 - f)(1 + 3g^{'}\cos\Theta) \quad (89)$$

where f is the fraction of scattering in the forward peak, $(1 + 3g^{'}\cos\Theta)$ are the first two terms of a Legendre polynomial expansion, and $g^{'}$ is a scaled asymmetry factor determined from the definition of g (i.e., $g = \int \cos\Theta d\Theta$). Hence, we have

$$g^{'} = \frac{(g - f)}{(1 - f)} \quad (90)$$

By requiring the 2nd moment of P_δ to be equal to that of the real phase function, which we approximate by a Henyey-Greenstein phase function, we have $f = g^2$ and hence

$$g^{'} = \frac{g}{1 + g} \quad (91)$$

We also need to scale the single scattering albedo and optical depth to account for the loss of the forward peak. Thus,

$$\omega_o^{'} = \frac{(1 - g^2)\omega_o}{1 - \omega_o g^2} \quad (92)$$

$$\tau^{'} = (1 - \omega_o g^2)\tau \quad (93)$$